Covariant Newtonian Gravity

$M = \mathbf{R}^4$

t_a covariant vector field that defines the time metric $\tau_{ab} := t_a t_b$ of signature $(0, 0, 0, +)$

h^{ab} contravariant tensor field of signature $(+, +, +, 0)$ that gives a space metric

∇_a derivative operator

$\nabla_a V^b := \partial_a V^b + \Gamma^a_{bc} V^c$

where Γ is the affine connection

Require:

(1) ∇_a is torsion free, i.e. for any scalar f,
$\nabla_a \nabla_b f = \nabla_b \nabla_a f \quad (\Rightarrow \Gamma^a_{bc} = \Gamma^a_{cb})$

(2) $\nabla_c h^{ab} = 0 = \nabla_c t_b$

(3) $h^{ab} t_b = 0$

Parallel transport of a vector V^b along a curve with tangent U^b:

$$U^a \nabla_a V^b = 0$$

In components:

$$U^a \partial_a V^b + U^a \Gamma^b_{ac} V^c = 0$$

With $U^a = \dfrac{dx^a}{d\lambda}$

$$\frac{dV^b}{d\lambda} + \frac{dx^a}{d\lambda} \Gamma^b_{ac} V^c = 0$$

And if $V^b = U^b$

$$\frac{d^2 x^b}{d\lambda^2} + \Gamma^b_{ac} \frac{dx^a}{d\lambda} \frac{dx^c}{d\lambda} = 0$$

- $\nabla_c t_b = 0$ plus M simply connected imply that there is a scalar field t such that $t_b = \partial_b t$. t is absolute time.

- h^{ab} defines a space metric on $t = const$ spaces. For a contravector V^a, define V_b by

$$h^{ab} V_b = V^a$$

If V_b satisfies this equation, so does

$$\hat{V}_b = V_b + t_b X$$

But if V^a is spacelike, i.e. $V^a t_a = 0$

$$\hat{V}_b = V_b$$

So for a spacelike V^a we can define the spatial length as

$$(h^{bc} V_b V_c)^{1/2}$$

- The time length of vector V^a is
  $(\tau_{ab} V^a V^b)^{1/2} = (t_a t_b V^a V^b)^{1/2}$

Consider a timelike curve parameterized by t so that the tangent vector ξ^a satisfies

$$t_a \xi^a = 1$$

Define the 4-acceleration by

$$A^a := \xi^b \nabla_b \xi^a$$

Find that

$$\begin{aligned} A^a t_a &= t_a \xi^b \nabla_b \xi^a = \xi^b \nabla_b (t_a \xi^a) \\ &= \xi^b \nabla_b (1) = 0 \end{aligned}$$

Thus, A^a is spacelike and has a well-defined norm. So we can speak of *the* magnitude of *the* spatial acceleration of a particle (don't have to ask "With respect to which frame?")

The 4-dim version of Newton's First Law is then

$$F^a = m A^a$$

where F^a must be a spacelike vector.

For a force derivable from a potential Φ

$$F^a := h^{ab} \partial_b \Phi$$

check that

$$F^a t_a = t_a h^{ab} \partial_b \Phi = 0$$

Newtonian gravity in field form

Covariant form of Poisson's equation

$$h^{ab}\nabla_a\nabla_b\Phi = 4\pi\rho$$

This will reduce to the familiar form of the equation in a global inertial system, whose existence we guarantee by requiring that the Riemann curvature tensor defined from Γ vanishes:

$$R^a_{bcd} = 0$$

If we use h^{ab} to raise indices we get

$$R^{abcd} = 0$$

This latter relation does not imply the former. The latter guarantees spatial flatness, i.e. that h^{ab} is Euclidean on the $t = const$ spaces.

Note that there is no spacetime metric. But we can create one if we add Absolute Space, a special reference from whose world lines are the integral curves of a distinguished timelike vector field $\tilde{V}^a$ such that $\nabla_b\tilde{V}^a$. Define

$$\eta^{ab} := h^{ab} - \tilde{V}^a\tilde{V}^b$$

Justification for introducing $\tilde{V}^a$: it is needed in formulating Maxwellian EM theory.

Problem: Attempts to detect which frame is Absolute Space fail. Easy route to STR: hold onto η^{ab} but acknowledge that there is no physically motivated way to split into a h^{ab} and a $\tilde{V}^a\tilde{V}^b$ piece. *Upshot*: We have the Minkowski metric!!!

Newton-Cartan

We retain absolute time, the space metric, and the time metric as "absolute objects". But the affine connection Γ becomes a dynamical object; and it is not flat in the presence of gravity.

- Generalized Poisson equation

  $R_{ab} := R^c_{abc} = 4\pi G \rho t_a t_b$

- Particles free falling in the gravitational field are geodesics

  $\xi^a \nabla_a \xi^b = 0$

As before, it is required that

(1) ∇_a is torsion free, i.e. for any scalar f,
$\nabla_a \nabla_b f = \nabla_b \nabla_a f \quad (\Rightarrow \Gamma^a_{bc} = \Gamma^a_{cb})$
(2) $\nabla_c h^{ab} = 0 = \nabla_c t_b$
(3) $h^{ab} t_b = 0$

But instead of requiring that curvature vanishes we require that

(4) $R^{ac}_{bd} = 0$

This provides a standard of rotation: parallel transport of a spacelike vector around a closed loop doesn't change it.

Show that standard Newtonian gravity can be recovered in the sense that there exists a scalar field Φ such that

$*$ The familiar Poisson equation $h^{ab}\nabla_a \nabla_b \Phi = 4\pi\rho$ holds

$*$ The affine connection Γ^a_{cb} can be split into an inertial part and a gravitational part

$$\Gamma^a_{cb} = {}^o\Gamma^a_{cb} + h^{ab}\nabla_b \Phi$$

where ${}^o\Gamma^a_{cb}$ is flat

$*$ The geodesic equation becomes

$$\frac{d^2 x^a}{d\lambda^2} + {}^o\Gamma^a_{bc}\frac{dx^b}{d\lambda}\frac{dx^c}{d\lambda} = -h^{ab}\nabla_b \Phi$$

Note: This split is not unique unless we live in an island universe and require that $\Phi \to 0$ as $r \to \infty$.